다음의 명령형 언어를 생각하자:

$$
\begin{array}{rcll}
e & ::= & n & \text{integer} \\
 & | & x & \text{variable} \\
 & | & e\,+\,e & \\
 & | & -e & \\
c & ::= & x := e & \text{assignment} \\
 & | & c\ ;\ c & \text{sequence} \\
 & | & \texttt{repeat}\ c\ \texttt{until}\ x & \text{repetition}
\end{array}
$$

"$\texttt{repeat}\ c\ \texttt{until}\ x$"는 변수 x가 양수가 될 때 까지 c를 반복한다.

분석의 목표는 변수가 가지는 정수가 홀수 인지 짝수인지를 분석하는 것이라고 하자.

프로그램 c의 의미는 $(\mathit{lfp} F)c$로 정의되고

$$
\begin{array}{lcl}
\multicolumn{3}{l}{F \in (\mathit{Cmd} \to \mathit{Mem} \to \mathit{Mem}) \to (\mathit{Cmd} \to \mathit{Mem} \to \mathit{Mem})} \\
\mathit{Mem} & = & \mathit{Var} \xrightarrow{\text{fin}} \mathit{Val} \\
\mathit{Val} & = & 2^{\mathbb{Z}}
\end{array}
$$

요약된 의미는 $(\mathit{lfp} \hat{F})c$로 정의된다

$$
\begin{array}{rcl}
 & & \hat{F} \in (\mathit{Cmd} \to \hat{\mathit{Mem}} \to \hat{\mathit{Mem}}) \to (\mathit{Cmd} \to \hat{\mathit{Mem}} \to \hat{\mathit{Mem}}) \\
\hat{\mathit{Mem}} & = & \mathit{Var} \xrightarrow{\text{fin}} \hat{\mathit{Val}} \\
\hat{\mathit{Val}} & = & \{\bot, \top, \mathtt{e}, \mathtt{o}\} \\
\hat{m}_1 \sqsubseteq \hat{m}_2 & \stackrel{\text{def}}{=} & \forall x \in \mathit{dom}(\hat{m}_1) \cup \mathit{dom}(\hat{m}_2) : \hat{m}_1\, x \sqsubseteq \hat{m}_2\, x \\
\forall x \notin \mathit{dom}(\hat{m}) : \hat{m}\, x & \stackrel{\text{def}}{=} & \bot_{\hat{\mathbb{Z}}}
\end{array}
$$

1. 갈로아 연결

$$\mathit{Val} \underset{\alpha_V}{\overset{\gamma_V}{\leftrightarrows}} \hat{\mathit{Val}}$$

은

$$
\alpha_V(X) = \begin{cases}
\bot & \text{if } X = \emptyset \\
\mathtt{e} & \text{if } \forall x \in X : x \text{ is even.} \\
\mathtt{o} & \text{if } \forall x \in X : x \text{ is odd.} \\
\top & \text{otherwise}
\end{cases}
$$

2. 갈로아 연결

$$(\mathit{Cmd} \to \mathit{Mem} \to \mathit{Mem}) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\mathit{Cmd} \to \hat{\mathit{Mem}} \to \hat{\mathit{Mem}})$$

은 부품 공간들 사이의 갈로아 연결

$$\mathit{Cmd} \underset{id}{\overset{id}{\leftrightarrows}} \mathit{Cmd} \qquad \mathit{Var} \underset{id}{\overset{id}{\leftrightarrows}} \mathit{Var} \qquad \mathit{Val} \underset{\alpha_V}{\overset{\gamma_V}{\leftrightarrows}} \hat{\mathit{Val}}$$

을 가지고 다음과 같이 정의한다:

$$\mathit{Mem} = \mathit{Var} \xrightarrow{\text{fin}} \mathit{Val} \underset{\alpha_M}{\overset{\gamma_M}{\leftrightarrows}} \mathit{Var} \xrightarrow{\text{fin}} \hat{\mathit{Val}} = \hat{\mathit{Mem}}$$

인 α_M는

$$\alpha_M(m) = \alpha_V \circ m \circ \gamma_{Var} = \alpha_V \circ m$$

이고

$$Mem \rightarrow Mem \underset{\alpha_C}{\overset{\gamma_C}{\leftrightarrows}} \hat{Mem} \rightarrow \hat{Mem}$$

인 α_C는

$$\alpha_C(x) = \alpha_M \circ x \circ \gamma_M$$

이며, α_M, α_C를 가지고 위의 α는

$$\alpha(\mathcal{C}) = \alpha_C \circ \mathcal{C} \circ \gamma_{Cmd} = \alpha_C \circ \mathcal{C}.$$

로 정의하면 갈로아 연결이 된다.

3. F와 $\hat{F}$를 정의하자.

$$\begin{array}{rcl} F\,\mathcal{C}\,(\texttt{repeat}\ c\ \texttt{until}\ x)\,m & = & ((\mathcal{C}\,c\,m)(x) \ni x \le 0\,?\,\mathcal{C}\,(\texttt{repeat}\ c\ \texttt{until}\ x)\,(\mathcal{C}\,c\,m)\ :\ \emptyset) \\ & \cup & ((\mathcal{C}\,c\,m)(x) \ni x > 0\,?\,\mathcal{C}\,c\,m\ :\ \emptyset) \\ \hat{F}\,\hat{\mathcal{C}}\,(\texttt{repeat}\ c\ \texttt{until}\ x)\,\hat{m} & = & (\hat{\mathcal{C}}\,(\texttt{repeat}\ c\ \texttt{until}\ x)\,(\hat{\mathcal{C}}\,c\,\hat{m})) \sqcup (\hat{\mathcal{C}}\,c\,\hat{m}) \end{array}$$

4. $\mathit{lfp}\hat{F}$가 $\mathit{lfp}F$의 안전한 요약임을 보이려면

$$\alpha(\mathit{lfp}F) \sqsubseteq \mathit{lfp}\hat{F}$$

임을 고정점 귀납법(*fixpoint induction*)으로 직접 증명하던가, Fixpoint Transfer 정리를 이용할 수 있으므로 다음을 증명하기만 하면 된다:

$$\alpha \circ F \sqsubseteq \hat{F} \circ \alpha.$$

즉,

$$\alpha(F\,\mathcal{C}) \sqsubseteq \hat{F}\,(\alpha\mathcal{C}).$$

모든 명령문에 대해서 위의 사실을 증명하면 되는데, 여기서는 "`repeat` c `until` x"에 대해서만 증명해 보자. $\mathtt{r} \overset{\text{let}}{=}$ "`repeat` c `until` x"로 하고 다시 쓰면, 증명할 것은, 임의의 $\hat{m}$에 대해서

$$\alpha(F\,\mathcal{C})\,\mathtt{r}\,\hat{m} \sqsubseteq \hat{F}\,(\alpha\mathcal{C})\,\mathtt{r}\,\hat{m}.$$

이제 좌변을 α의 성질을 이용해서 전개해 보면 우변보다 작거나 같게 된다:

$$\begin{array}{rcll} \alpha(F\,\mathcal{C})\,\mathtt{r}\,\hat{m} & = & (\alpha_C \circ (F\,\mathcal{C}))\,\mathtt{r}\,\hat{m} & \text{(by def. of } \alpha) \\ & = & (\alpha_C(F\,\mathcal{C}\,\mathtt{r}))\,\hat{m} & \\ & = & (\alpha_M \circ (F\,\mathcal{C}\,\mathtt{r}) \circ \gamma_M)\,\hat{m} & \text{(by def. of } \alpha_C) \\ & = & \alpha_M(F\,\mathcal{C}\,\mathtt{r}\,(\gamma_M\hat{m})) & \\ & \sqsubseteq & \alpha_M(\mathcal{C}\,\mathtt{r}\,(\mathcal{C}\,c\,(\gamma_M\hat{m}))\ \cup\ \mathcal{C}\,c\,(\gamma_M\hat{m})) & \text{by def. of } F \\ & = & \alpha_M(\mathcal{C}\,\mathtt{r}\,(\mathcal{C}\,c\,(\gamma_M\hat{m})))\ \sqcup\ \alpha_M(\mathcal{C}\,c\,(\gamma_M\hat{m})) & \text{by Galois conn.} \\ & \sqsubseteq & (\alpha\mathcal{C})\,\mathtt{r}\,(\alpha_M(\mathcal{C}\,c\,(\gamma_M\hat{m})))\ \sqcup\ (\alpha\mathcal{C})\,c\,(\alpha_M(\gamma_M\hat{m})) & \text{by Prop.1 twice} \\ & \sqsubseteq & (\alpha\mathcal{C})\,\mathtt{r}\,((\alpha\mathcal{C})\,c\,(\alpha_M(\gamma_M\hat{m})))\ \sqcup\ (\alpha\mathcal{C})\,c\,(\alpha_M(\gamma_M\hat{m})) & \text{by Prop.1 twice} \\ & \sqsubseteq & (\alpha\mathcal{C})\,\mathtt{r}\,((\alpha\mathcal{C})\,c\,\hat{m})\ \sqcup\ (\alpha\mathcal{C})\,c\,\hat{m} & \text{by Galois conn.} \\ & = & \hat{F}\,(\alpha\mathcal{C})\,\mathtt{r}\,\hat{m}. & \end{array}$$

Proposition 1 $f \in A \rightarrow B$는 단조함수이고 $A \underset{\alpha_A}{\overset{\gamma_A}{\leftrightarrows}} \hat{A}$이고 $B \underset{\alpha_B}{\overset{\gamma_B}{\leftrightarrows}} \hat{B}$일 때,

$$\alpha_B(f\,a) \sqsubseteq (\alpha_{A \rightarrow B} f)(\alpha_A a)$$

이다. 왜냐하면,

$$\begin{array}{rcl} \alpha_B(f\,a) & \sqsubseteq & \alpha_B(f\,(\gamma_A(\alpha_A a))) \\ & = & (\alpha_B \circ f \circ \gamma_A)(\alpha_A a) \\ & \sqsubseteq & (\alpha_{A \rightarrow B} f)(\alpha_A a). \end{array}$$